Indian Statistical Institute

Mid-Semestral Examination 2011-2012

M.Math First Year

Analysis of Several Variables

Time : 3 Hours Date : 23.09.2011 Maximum Marks : 100 Instructor : Jaydeb Sarkar

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Answer all questions.

Q1. (10 marks) Let E be a non-compact subset of $\mathbb{R}^n$. Prove that there exists an unbounded continuous function $f : E \to \mathbb{R}$.

Q2. (10 marks) Let U be an open subset of $\mathbb{R}^2$ and $f : U \to \mathbb{R}$. Show that $D_{12}f = D_{21}f$ on U if $D_{12}f$ and $D_{21}f$ are continuous functions on U.

Q3. (20 marks) Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation. Prove that

$$v(T(U)) = |\det T| v(U),$$

where U is a box in $\mathbb{R}^n$ and v is the volume function.

Q4. (20 marks) If $f_1(x, y, z) = xy + yz + xz$ and $f_2(x, y, z) = x + e^y \sin z - 1$, prove that there exists differentiable functions $x(z)$ and $y(z)$ in a neighborhood of $z = 0$ such that $x(0) = 1$ and $y(0) = 0$ and

$$f_1(x(z), y(z), z) = 0 = f_2(x(z), y(z), z).$$

Compute $x'(0)$ and $y'(0)$.

Q5. (20 marks) Let $\| \cdot \|_2$ be the Euclidean norm on $\mathbb{R}^n$ (that is, $\|x\|_2 = \|(x_1, \ldots, x_n)\|_2 = (\sum_{i=1}^n x_i^2)^{\frac{1}{2}}$) and $\| \cdot \|$ be any other norm. Prove that there exists constants $\alpha, \beta > 0$ such that

$$\alpha \|x\|_2 \leq \|x\| \leq \beta \|x\|_2.$$

Q6. (10 marks) Let $f : \mathbb{R}^n \to \mathbb{R}$ be a C^2 map and $A = (a_{ij})_{n \times n}$ be an orthogonal matrix (that is, $AA^t = I$). Prove that $(\bigtriangledown^2 f)(Ax) = (\bigtriangledown^2 (fA))(x)$ for all $x \in \mathbb{R}^n$.
[Note: For a C^2 map f, $\bigtriangledown^2 f = \sum_{i=1}^n D_{ii} f$ is called the Laplacian of f.]

Q7. (10 marks) Let $T : \mathbb{R}^n \to \mathbb{R}$ be a linear map. Prove that there exists a unique $y \in \mathbb{R}^n$ such that

$$T(x) = x \cdot y \qquad (\forall x \in \mathbb{R}^n)$$

and $\|T\| = \|y\|$.